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Matula numbers, Gödel numbering and Fock space

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Abstract By making use of Matula numbers, which give a 1-1 correspondence between rooted trees and natural numbers, and a Gödel type relabelling of quantum states, we construct a bijection between rooted trees and vectors in the Fock space. As a by product of the aforementioned correspondence (rooted trees $\leftrightarrow$ Fock space) we show that the fundamental theorem of arithmetic is related to the grafting operator, a basic construction in many Hopf algebras. Also, we introduce the Heisenberg–Weyl algebra built in the vector space of rooted trees rather than the usual Fock space. This work is a cross-fertilization of concepts from combinatorics (Matula numbers), number theory (Gödel numbering) and quantum mechanics (Fock space).

Keywords Matula numbers · Rooted trees · Hopf algebra · Gödel relabelling · Heisenberg–Weyl algebra · Fock space

Mathematics Subject Classification (2010) 05C05 · 11A41 · 81V70

1 Introduction

The Matula (M) numbers [1,2], which assign a 1-1 correspondence between rooted trees and natural numbers, were introduced back in 1968 in a short abstract [1] and have found applications mainly in the context of chemistry, e.g., various properties of rooted trees like topological indices are listed in [3] as a function of the M numbers. See also [4–7]. Matula ended his definition pointing out: “Certain interesting relationships between the theories of primes and trees will be developed utilizing this ‘natural’ correspondence.” This work points towards giving a support to this affirmation in the

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context of posterior developments on Hopf algebras [8–26] and quantum mechanics [27–29].

Graphical methods, e.g., trees (rooted, planar, etc), are recurrent objects in many algebraic constructions underlying important developments, such as, the Grossman–Larson (G–L) Hopf algebra [8–10] to compute certain differential operators, the celebrated Connes–Kreimer (C–K) Hopf algebra of renormalization in quantum field theory [11–13], C–K Hopf algebra of planar decorated rooted trees [14–16], the connection between G–L and C–K Hopf algebras [17,18], the interplay of Runge–Kutta methods and renormalization [19–21], QED [22], many-body calculations [23–25], Butcher group in numerical analysis [30,31], Heisenberg–Weyl algebra [32–35], etc. Therefore, due to the broad interest in constructions involving graphical methods, it is clear that every possible way of re-expressing them is worth of consideration, specially, if it unravels connections between apparent dissimilar areas of general interest, like combinatorics, number theory and quantum mechanics.

In this work, based upon Matula numbers [1] and a Gödel type relabelling of quantum states due to Spector [27–29], we give a 1-1 correspondence between rooted trees and Fock space vectors. With a view towards establishing connections we revisit two fundamental concepts: the grafting operator (see, e.g., [19,20]), a common construction in many Hopf algebras, and the Heisenberg–Weyl algebra (see, e.g., [32–35] and references therein) in the realm of quantum many-body theory. More precisely, we introduce the counterparts of the grafting operator in Fock space and the Heisenberg–Weyl algebra in the space with basis the rooted trees.

In the context of chemistry, we recall that the Heisenberg–Weyl algebra is a recurrent and important construction, e.g., in the context of the occupation number representation of state vectors in the formalism of second quantization describing a quantum system of identical bosonic particles (see, e.g., Ref. [36]), in the formulations of chemical reactions based on the second quantization formalism (for a review see Ref. [37]), etc. We introduce here creation and annihilation operators acting on rooted trees that are the direct counterparts of the usual bosonic creation and annihilation operators (the basic building blocks of the Heisenberg–Weyl algebra). In the occupation number representation, a generic Fock state vector of a many-body bosonic system is constructed by the action of creation operators on the vacuum state of the underlying physical Hilbert space (see, e.g., the lattice model considered in Ref. [29] or the bosonic quantum system made of indistinguishable particles of, e.g., Refs. [38,39]) and we show here that this is nothing more than the grafting operator acting on rooted trees. Therefore, our results establish new representations (and consequent re-interpretations) of recurrent concepts in quantum chemistry which are at the same time of basic relevance and of a fundamental nature. Our main objective here is to establish a first step towards the construction of a bridge between the methods of many-body quantum theory, a ubiquitous theme in quantum chemistry, and the structure of some important Hopf algebras in a manner that a direct connection with concepts of number theory can be pursued. The aforementioned bridge opens the way to tackle directly problems in many-body quantum theory from the point of view of algebraic constructions based on rooted trees (like the Hopf algebras mentioned above) and vice versa.

This work is organized as follows. In Sect. 2 we introduce the basic constructions on rooted trees: Matula number and the grafting operator. In Sect. 3 we review briefly the Gödel type relabelling of quantum states due to Spector [27–29]. In Sect. 4 we construct the grafting operator in Fock space and the Heisenberg–Weyl algebra on rooted trees. Finally, in Sect. 5 we make some concluding remarks.

2 Matula numbers and the grafting operator

We begin introducing some fundamental definitions and fixing the notation. From this moment on we establish the following convention, whenever we write tree we refer to rooted tree. We start with

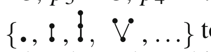
Definition 1 A rooted tree τ is a finite graph, connected and without loops, with a special vertex v called the root r, i.e., $v = r$ which has only outgoing edges.

E.g., we have

$\tau =$  . (1)

Definition 2 The degree d (or fertility) of the root vertex $v = r$ is defined as the number of outgoing edges from r.

E.g., the tree in Eq. (1) has $d = 3$.

We write the prime numbers in ascending order: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, and so on. We set $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$, $\mathbb{T} =$  to be the set of the primes, natural numbers and rooted trees, respectively. Throughout this work we use the conventions $p_i, p_j, p_k \in \mathbb{P}$; $a, b, i, j, k \in \mathbb{N}$; $\tau, \tau', \tau_k, \tau'_k \in \mathbb{T}$. Every natural number a, b, n, etc can be written as $a = \prod_{p_j|a} p_j^{\alpha_j}$, $b = \prod_{p_j|b} p_j^{\beta_j}$, $n = \prod_{p_j|n} p_j^{\nu_j}$, etc, as a direct consequence of the fundamental theorem of arithmetic.

We define the map

$$\begin{aligned} p : \mathbb{N} &\to \mathbb{P} \\ k &\mapsto p(k) = p_k \end{aligned}$$

and the inverse map

$$\begin{aligned} p^{-1} : \mathbb{P} &\to \mathbb{N} \\ p_k &\mapsto p^{-1}(p_k) = k \end{aligned}.$$

In words, the map p assigns to $k \in \mathbb{N}$ the prime number with index k, i.e., p_k. The inverse map p^{-1} identifies the index of the prime in which it acts.

Let us next define the M numbers following Ref. [2], which established a 1-1 correspondence between natural numbers and rooted trees.

Definition 3 Let τ be a rooted tree with root r. If τ has only one vertex, then $n(\tau) = n(\bullet) = 1$. If the number of vertices of τ is greater than 1, then we assign a number $n(\tau) \in \mathbb{N}$ to each rooted tree by

$$n(\tau) = \prod_{i=1}^{d} p(n(\tau_i)), \tag{2}$$

where $\tau_i, i = 1, \ldots, d$ are connected components (there is a path that connects any two vertices of τ_i) of $\tau - r$. $n(\tau)$ is called the Matula (M) number of τ.

The inverse of the M number is the unique tree $\tau(n)$ associated to a given natural number n.

Let us assign the M number to some simple trees. We will show that $n($$) = 2$ and $n\left(\right.$$\left.\right) = 7$. We have

$n($$) = p(n(\bullet)) = p(1) = 2.$

Also,

$n\left(\right.$$\left.\right) = p(n($$))$
$= p(p(n(\bullet)) \times p(n(\bullet)))$
$= p(p(1) \times p(1)) = p(2 \times 2) = p(4) = 7.$

A list of M numbers up to 45 is given in Ref [2]. See also [40] for an alternative definition of M numbers closely related to our observations (1) and (2) below.

For our purposes the basic observations, directly related to Definition 3 above, are important:

(1) All prime numbers have a tree representation such that $d = 1$. Indeed, suppose on the contrary that $d > 1$, then we have a product of primes by Eq. (2) with at least two factors, which is an absurd. Therefore, we must have $d = 1$ for all τ obeying $n(\tau) = p \in \mathbb{P}$.

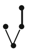
We introduce the set $\mathfrak{T} = \{$, , , , $\ldots\} \subset \mathbb{T}$ such that $d = 1$, i.e., all trees with roots of degree 1. Due to the observation (1) above we call all trees with $d = 1$ prime trees (sometimes referred as primitive trees) and, as a consequence, $\mathfrak{T}$ is the set of all prime trees. We associate each p_k with the rooted tree $\pi_{\tau(k)}$, e.g., $p_4 = 7 \cong \pi_{\tau(4)} = \pi$ $=$ .

(2) There is a natural operation on the set of rooted trees $\mathbb{T}$ called tree multiplication $*$

$$\begin{aligned} * : \mathbb{T} \times \mathbb{T} &\to \mathbb{T} \\ (\tau_1, \tau_2) &\mapsto \tau_1 * \tau_2 = \tau, \end{aligned}$$

where

$$\tau = \{\text{the new tree with } d = d_1 + d_2 \text{ formed from } \tau_1 \text{ and } \tau_2 \text{ by putting together their roots } r_1 \text{ and } r_2\}.$$

E.g., for $\tau_1 =$ and $\tau_2 =$ we have $\tau_1 * \tau_2 =$  $= \tau_2 * \tau_1$. The commutative property of the operation $*$ is a reflection of the fact that we are using rooted trees rather than planar trees. We write $\tau^k = \tau * \cdots * \tau$ (τ appearing k times). At this stage we naturally recognize a homomorphism between multiplication on trees ($*$) and multiplication on natural numbers ($\times$). Indeed, setting $\tau = \tau_1 * \cdots * \tau_d$, $\tau' = \tau'_1 * \cdots * \tau'_{d'}$, $\hat{\tau}_i = \tau_i$ if $i = 1, \ldots, d$ and $\hat{\tau}_i = \tau'_{i-d}$ if $i = d+1, \ldots, d+d'$ we have

$$\begin{aligned} n(\tau * \tau') &= \prod_{i=1}^{d+d'} p(n(\hat{\tau}_i)) \\ &= \prod_{i=1}^{d} p(n(\tau_i)) \times \prod_{j=1}^{d'} p(n(\tau'_j)) \\ &= n(\tau) \times n(\tau'). \end{aligned}$$

The identity here is $\tau = \bullet$.

Of course, since there is a bijection between natural numbers and trees, on rooted trees we have counterparts of the maps p and p^{-1} defined in $\mathbb{N}$ and $\mathbb{P}$, respectively. We define the maps

$$\begin{aligned} \pi &: \mathbb{T} \to \mathfrak{T} \\ &\tau \mapsto \pi(\tau) = \pi_\tau \end{aligned}$$

where $\pi_\tau =$ {tree obtained from τ by attaching and edge to τ such that one vertex is coincident with the root r of τ and the other vertex is the new root}. E.g., we have

$\pi\Big(\tau =$  $\Big) =$ .

Note that, as follows from the definition, $\pi_\tau = \pi(\tau)$ is a prime tree. The inverse map with respect to π is given by

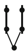
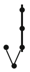
$$\begin{aligned} \pi^{-1} &: \mathfrak{T} \to \mathbb{T} \\ &\pi_\tau \mapsto \pi^{-1}(\pi_\tau) = \tau \end{aligned}$$

where τ = {tree obtained from π_τ by deleting the outgoing edge from the root}. Note that the degree of the root of $\pi^{-1}(\pi_\tau) = \tau$ is such that $d \geq 1$. E.g., we have

$$\pi^{-1}\left(\pi_\tau = \right.$$  $$\left.\right) = $$  .

The definitions of the maps p, p^{-1} and their counterparts π, π^{-1} are crucial for the definitions to come.

Now we establish the underlying vector space with basis the rooted trees and the associated inner product. We follow closely the notation of Ref. [17], except that we place the root of a given tree at the bottom rather than at the top. We let $|\tau|$ to be the number of vertices of τ. We define $\mathbb{T}_n = \{\tau \in \mathbb{T} : |\tau| = n + 1\}$ and write $k\{\mathbb{T}\} = \oplus_{n\geq 0} k\{\mathbb{T}_n\}$ for the graded vector space over a field k of characteristic zero (here we implicit assume this field to be $\mathbb{R}$ = the real numbers) with basis consisting of elements belonging to $\mathbb{T}$. Also, we let H_K (the subscript K here stands for Kreimer) to be the vector space built on monomials of rooted trees, i.e., forests (see Ref. [17]). Each $k\{\mathbb{T}_n\}$ is itself a vector space. A typical element of $k\{\mathbb{T}_4\}$ is

$$-$$  $$+\frac{1}{5}$$  $$+\sqrt{2}$$  $$\in k\{\mathbb{T}_4\}. \quad (3)$$

Following [17] we can endow $k\{\mathbb{T}\}$ with an inner product as follows

$$(\tau, \tau') = |SG(\tau)|\delta_{\tau,\tau'}, \quad (4)$$

where δ is the Kronecker delta and SG is the symmetry group of τ (see Ref. [17] for a precise definition). We define a linear map $B_+ : H_K \to k\{\mathbb{T}\}$

$$B_+ : (\tau_1, \ldots, \tau_d) \to B_+(\tau_1, \ldots, \tau_d) = \tau. \quad (5)$$

or, equivalently,

$$\begin{aligned} B_+(\tau_1, \ldots, \tau_d) &= \pi(\tau_1) * \cdots * \pi(\tau_d) \\ &= \pi_{\tau_1} * \cdots * \pi_{\tau_d}. \end{aligned} \quad (6)$$

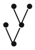
The operator B_+ is sometimes known as the grafting operator. The inverse map can be written as

$$B_- : \tau \to B_-(\tau) = (\tau_1, \ldots, \tau_d) \quad (7)$$

or, equivalently,

$$B_-(\tau) = \left(\pi^{-1}\left(\pi_{\tau_1}\right), \ldots, \pi^{-1}\left(\pi_{\tau_d}\right)\right) = (\tau_1, \ldots, \tau_d). \quad (8)$$

Let us work out an example: take $\tau_1 = \bullet$ and $\tau_2 =$  to get

$$B_+(\bullet,$$  $$) =$$  (9)

and

$$B_-($$  $$) = (\bullet,$$  $$).$$ (10)

3 Gödel relabelling of quantum states due to Spector

To establish a connection with the structure presented above we now review briefly the Göldel relabelling of quantum states due to Spector [27–29]. We will follow the quantum system described in Ref. [29]. The construction here can be applied directly to describe a bosonic quantum system made of identical particles in the second quantization formalism. In this case, the relevant space is the bosonic Fock space = direct sum of symmetrized tensor copies of $L^2(\mathbb{R}^3)$ (see Ref. [36]). In the occupation number formalism for a bosonic system each vector in Fock space is represented by

$$|\alpha_1, \alpha_2, \alpha_3, \ldots\rangle \in \mathscr{H} = \otimes_{k=1}^{\infty} H_k$$

with $|\alpha_j\rangle \in H_j$ and $\alpha_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ meaning the excitation associated with subsystem j. The inner product in $\mathscr{H}$ is given by $\langle\alpha_1, \alpha_2, \ldots|\beta_1, \beta_2, \ldots\rangle = \langle\alpha_1|\beta_1\rangle\langle\alpha_2|\beta_2\rangle\cdots$ with $\hat{O}_1\hat{O}_2\cdots|\alpha_1\rangle|\alpha_2\rangle\cdots = \hat{O}_1|\alpha_1\rangle\hat{O}_2|\alpha_2\rangle\cdots$ where $\hat{O}_j$ acts in H_j. Each H_j ($j \in \mathbb{N} \equiv \{1, 2, \ldots\}$) is the Hilbert space associated with a single degree of freedom and described by a quantum harmonic oscillator. We write $|\alpha_j\rangle \in H_{\text{osc}} = H_j \; \forall j$ for the eigenstate of the quantum mechanical harmonic oscillator, i.e., $\hat{H}_j = \hat{N}_j + 1/2 = \hat{a}_j^{\dagger}\hat{a}_j + 1/2$, $[\hat{a}_i, \hat{a}_j^{\dagger}] = \delta_{i,j}$ and $|\alpha_j\rangle = (\hat{a}_j^{\dagger})^{\alpha_j}|0\rangle/\sqrt{\alpha_j!}$. All the other commutators involving $\hat{a}_i$ and $\hat{a}_i^{\dagger}$ being zero: $[\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^{\dagger}, \hat{a}_j^{\dagger}] \; \forall i, j$. The algebra generated by $\{1, \hat{a}_j^{\dagger}, \hat{a}_j, \hat{N}_j\}$ is known as the Heisenberg–Weyl algebra. It is a direct consequence of the commutation relations between $\hat{a}_j^{\dagger}$ and $\hat{a}_j$ that the following relations hold

$$\hat{a}_j^{\dagger}|\alpha_j\rangle = \sqrt{\alpha_j + 1}|\alpha_j + 1\rangle \tag{11}$$

and

$$\hat{a}_j|\alpha_j\rangle = \sqrt{\alpha_j}|\alpha_j - 1\rangle. \tag{12}$$

The basic ingredient behind the Gödel relabelling of quantum states due to Spector [27–29] is the identification of each quantum state in $|a\rangle \in \mathscr{H}$ with the factorization of each $a \in \mathbb{N}$ as powers of prime numbers. More precisely, given $\mathbb{N} \ni a = p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \cdots = 2^{\alpha_1} \times 3^{\alpha_2} \times 5^{\alpha_3} \times \cdots$ we will identify $p_j \mapsto$ orbital in position j and $\alpha_j \mapsto |\alpha_j\rangle \in H_j$.

In Dirac ket notation we make the identification

$$\begin{aligned}|a\rangle &= |2^{\alpha_1} \times 3^{\alpha_2} \times 5^{\alpha_3} \times \cdots\rangle \\ &= |\alpha_1, \alpha_2, \alpha_3, \ldots\rangle \\ &= \frac{\left(\hat{a}_1^\dagger\right)^{\alpha_1}}{\sqrt{\alpha_1!}} \frac{\left(\hat{a}_2^\dagger\right)^{\alpha_2}}{\sqrt{\alpha_2!}} \frac{\left(\hat{a}_3^\dagger\right)^{\alpha_3}}{\sqrt{\alpha_3!}} \ldots |\text{vac}\rangle,\end{aligned} \tag{13}$$

where $|\text{vac}\rangle$ is the vacuum of the Fock space. $(\hat{a}_i^\dagger)^{\alpha_i}$ means $\hat{a}_i^\dagger$ acting α_i times in the orbital located at i. The first equality in Eq. (13) follows from the usual factorization of a natural number in terms of prime factors; the second equality follows from the quantum mechanical representation of $|a\rangle$ with $a \in \mathbb{N}$; the third equality follows from the definition of the operator $\hat{a}_j^\dagger$. E.g., take $|a\rangle = |1\rangle$, $|b\rangle = |2\rangle$ and $|n\rangle = |60\rangle$, so that we have:

$$\begin{aligned}|a\rangle = |1\rangle &= |2^{\alpha_1=0} \times 3^{\alpha_2=0} \times 5^{\alpha_3=0} \times 7^{\alpha_4=0} \times \cdots\rangle \\ &= |0, 0, 0, 0, \ldots\rangle \\ &= |\text{vac}\rangle, \\ |b\rangle = |2\rangle &= |2^{\beta_1=1} \times 3^{\beta_2=0} \times 5^{\beta_3=0} \times 7^{\beta_4=0} \times \cdots\rangle \\ &= |1, 0, 0, 0, \ldots\rangle \\ &= \left(\hat{a}_1^\dagger\right)^{\beta_1=1} |\text{vac}\rangle\end{aligned}$$

and

$$\begin{aligned}|n\rangle = |60\rangle &= |2^{\nu_1=2} \times 3^{\nu_2=1} \times 5^{\nu_3=1} \times 7^{\nu_4=0} \times \cdots\rangle \\ &= |2, 1, 1, 0, \ldots\rangle \\ &= \frac{\left(\hat{a}_1^\dagger\right)^{\nu_1=2}}{\sqrt{2!}} \left(\hat{a}_2^\dagger\right)^{\nu_2=1} \left(\hat{a}_3^\dagger\right)^{\nu_3=1} |\text{vac}\rangle.\end{aligned}$$

Note that the Gödel relabelling of quantum states due to Spector was crucial here since it maps each Fock space vector into a state indexed by a natural number and the connection of natural numbers with rooted trees is given by the M numbers.

Setting ($\tau(n)$ is the inverse of the M number)

$$|n) = |SG(\tau(n))|^{1/2}|n\rangle$$

we obtain

$$(n|n') = |SG(\tau(n))|^{1/2}|SG(\tau(n'))|^{1/2}\langle n|n'\rangle = |SG(\tau(n))|\delta_{n,n'}, \tag{14}$$

where we have used $\langle a|b\rangle = \delta_{a,b}$. Equation (14) is the equivalent of the inner product in Eq. (4) on rooted trees.

In order to get used with the correspondence rooted trees $\leftrightarrow$ occupation number formalism we give some examples. We have:

$\tau =$ $\cong |n\big($  $\big)\rangle = |3\rangle = \hat{a}_2^{\dagger}|\text{vac}\rangle$ and $\tau' =$  $\cong |n\big($  $\big)\rangle = |7\rangle = \hat{a}_4^{\dagger}|\text{vac}\rangle.$

The equivalent of Eq. (3) is

$$-|9\rangle + \frac{1}{5}|10\rangle + \sqrt{2}|17\rangle = -\frac{\left(\hat{a}_2^{\dagger}\right)^2}{\sqrt{2!}}|\text{vac}\rangle + \frac{1}{5}\hat{a}_1^{\dagger}\hat{a}_3^{\dagger}|\text{vac}\rangle + \sqrt{2}\hat{a}_7^{\dagger}|\text{vac}\rangle \in \mathscr{H}. \quad (15)$$

Schematically, our procedure so far can be best summarized by the diagrams below, which establish a 1-1 correspondence between vectors in $k\{\mathbb{T}\}$ and $\mathscr{H}$

(1) From rooted trees to Fock space vectors

$$\tau \xrightarrow{\text{Matula}} |a(\tau) = 2^{\alpha_1} \times 3^{\alpha_2} \times 5^{\alpha_3} \times \cdots\rangle \xrightarrow{\text{Spector}} \frac{\left(\hat{a}_1^{\dagger}\right)^{\alpha_1}}{\sqrt{\alpha_1!}} \frac{\left(\hat{a}_2^{\dagger}\right)^{\alpha_2}}{\sqrt{\alpha_2!}} \frac{\left(\hat{a}_3^{\dagger}\right)^{\alpha_3}}{\sqrt{\alpha_3!}} \cdots |\text{vac}\rangle. \quad (16)$$

(2) From Fock space vectors to rooted trees

$$\frac{\left(\hat{a}_1^{\dagger}\right)^{\alpha_1}}{\sqrt{\alpha_1!}} \frac{\left(\hat{a}_2^{\dagger}\right)^{\alpha_2}}{\sqrt{\alpha_2!}} \frac{\left(\hat{a}_3^{\dagger}\right)^{\alpha_3}}{\sqrt{\alpha_3!}} \cdots |\text{vac}\rangle \xrightarrow{\text{Spector}} |a = 2^{\alpha_1} \times 3^{\alpha_2} \times 5^{\alpha_3} \times \cdots\rangle \xrightarrow{\text{Matula}} \tau(a). \quad (17)$$

4 Grafting operator in Fock space and Heisenberg–Weyl algebra on rooted trees

From now on we write $|j_1, \ldots, j_d\rangle \in H_K$. We set $n = \prod_{p_j|n} p_j^{\nu_j} = p_{j_1}^{\nu_{j_1}} \times \cdots \times p_{j_k}^{\nu_{j_k}} = p_{j_1} \times \cdots \times p_{j_d}$ (ν_{j_l} counts the number of times p_{j_l} appears in $p_{j_1} \times \cdots \times p_{j_d}$), then $k \equiv \sum_{j:p_j|n} 1$ and $d \equiv \sum_{j:p_j|n} \nu_j$. E.g., take $n = 20 = 2^2 \times 5 = (p_{j_1} = 2) \times (p_{j_2} = 2) \times (p_{j_3} = 5)$ with $j_1 = 1 = j_2$ and $j_3 = 3$, then $k = 2$ and $d = 3$. Although the construction that follows below does not make reference to trees we still use the index d (recall that we use d for the degree of the root r of $\tau \cong |n\rangle$) in $n = p_{j_1} \times \cdots \times p_{j_d}$ for clearness, i.e., to enhance the connection with the constructions of the B_+ and B_- for rooted trees. In this way, motivated by the definitions of B_+ and B_- in Eqs. (5), (6), we define the linear map $\hat{B}_+ : H_K \to \mathscr{H}$

$$\hat{B}_+ : |j_1, \ldots, j_d\rangle \to \hat{B}_+|j_1, \ldots, j_d\rangle = |n\rangle \quad (18)$$

or

$$\hat{B}_+|j_1, \ldots, j_d\rangle = |p(j_1) \times \cdots \times p(j_d)\rangle$$

$$
\begin{aligned}
&= |p_{j_1} \times \cdots \times p_{j_d}\rangle \\
&= |p_{j_1}^{\nu_{j_1}} \times \cdots \times p_{j_k}^{\nu_{j_k}}\rangle \\
&= \frac{\left(\hat{a}_{j_1}^{\dagger}\right)^{\nu_{j_1}}}{\sqrt{\nu_{j_1}!}} \cdots \frac{\left(\hat{a}_{j_k}^{\dagger}\right)^{\nu_{j_k}}}{\sqrt{\nu_{j_k}!}} |\text{vac}\rangle \\
&= |n\rangle. \qquad (19)
\end{aligned}
$$

The first and second equalities are the analogous of the equalities in Eq. (6) using the correspondence $\pi \rightarrow p, \tau \rightarrow j$ and $* \rightarrow \times$. To obtain the last two lines we have used Spector relabelling of quantum states of Eq. (13). The inverse map is

$$\hat{B}_{-} : |n\rangle \rightarrow \hat{B}_{-}|n\rangle = |j_1, \ldots, j_d\rangle \qquad (20)$$

or

$$\hat{B}_{-}|n\rangle = |p^{-1}(p_{j_1}), \ldots, p^{-1}(p_{j_d})\rangle = |j_1, \ldots, j_d\rangle, \qquad (21)$$

which are the analogous of Eqs. (7) and (8), respectively. It is interesting to interpret the maps in Eqs. (19) and (21) from the point of view of Number Theory (NT) and many body quantum theory. Given $j_1, \ldots, j_d$, $\hat{B}_{+}$ selects the vector of the Fock space ν_{j_1} times excited in orbital j_1, ν_{j_2} times excited in orbital j_2, and so on until arrive at orbital j_k, or, using Spector identification, we could say that the resulting state $|n\rangle$ corresponds to the product of the prime factors with indices $j_1, \ldots, j_d$. Given $|n\rangle = |\prod_{p_j|n} p_j^{\nu_j}\rangle = |p_{j_1} \times \ldots \times p_{j_d}\rangle$ with $d \equiv \sum_{j:p_j|n} \nu_j$, $\hat{B}_{-}$ selects the monomials composed of the indices j in p_j in the prime number decomposition of n. The identifications above are very compelling from the point of view of the interplay of NT and many-body theory, because they are related to the fundamental theorem of arithmetic and admit a corresponding occupation number interpretation. Let us work out an example: take $j_1 = 1$ and $j_2 = 4$ to get

$$
\begin{aligned}
\hat{B}_{+}|j_1, j_2\rangle = \hat{B}_{+}|1, 4\rangle &= |p(1) \times p(4)\rangle \\
&= |p_1 \times p_4\rangle \\
&= \hat{a}_1^{\dagger}\hat{a}_4^{\dagger}|\text{vac}\rangle \\
&= |14\rangle. \qquad (22)
\end{aligned}
$$

Also, since $n = 14 = 2 \times 7 = p_1^{\nu_1=1} \times p_4^{\nu_4=1}$, we have

$$\hat{B}_{-}|n\rangle = \hat{B}_{-}|14\rangle = |p^{-1}(p_1), p^{-1}(p_4)\rangle = |1, 4\rangle \qquad (23)$$

with $|n\rangle = |14\rangle \cong$ . Equations (22) and (23) are the analogous of Eqs. (9) and (10).

Now we show how to re-interpret the operators $\hat{a}_j^{\dagger}$ and $\hat{a}_j$ by considering their action on trees instead of the usual Fock space. We refer to $\hat{\alpha}_{\tau}^{\dagger}$ and $\hat{\alpha}_{\tau}$ as the counterparts of

$\hat{a}_j^\dagger$ and $\hat{a}_j$, respectively. For this purpose, we calculate the action of $\hat{a}_j^\dagger$ and $\hat{a}_j$ in $|n\rangle$ in a way that a direct correspondence with rooted trees can be pursued. We have:

$$\begin{aligned}\hat{a}_{j_l}^\dagger|n = p_{j_1} \times \cdots \times p_{j_d} = p_{j_1}^{\nu_{j_1}} \times \cdots \times p_{j_k}^{\nu_{j_k}}\rangle \\ = \sqrt{\nu_{j_l}+1}|\ldots, \nu_{j_1}, \ldots, \nu_{j_l}+1, \ldots, \nu_{j_k}\cdots\rangle \\ = \sqrt{\nu_{j_l}+1}|p_{j_1}^{\nu_{j_1}} \times \cdots \times p_{j_l}^{\nu_{j_l}+1} \times \cdots \times p_{j_k}^{\nu_{j_k}}\rangle \\ = \sqrt{\nu_{j_l}+1}|n \times p_{j_l}\rangle \end{aligned} \tag{24}$$

where we have used Eq. (11) for the first equality, the Gödel relabelling of quantum states due to Spector in Eq. (13) for the first and second equalities and last line is a direct consequence of the definition $n = p_{j_1}^{\nu_{j_1}} \times \cdots \times p_{j_k}^{\nu_{j_k}}$. In a similar way we can show

$$\begin{aligned}\hat{a}_{j_l}|n = p_{j_1} \times \cdots \times p_{j_d} = p_{j_1}^{\nu_{j_1}} \times \cdots \times p_{j_k}^{\nu_{j_k}}\rangle \\ = \sqrt{\nu_{j_l}}|\ldots, \nu_{j_1}, \ldots, \nu_{j_l}-1, \ldots, \nu_{j_k}\ldots\rangle \\ = \sqrt{\nu_{j_l}}|p_{j_1}^{\nu_{j_1}} \times \cdots \times p_{j_l}^{\nu_{j_l}-1} \times \cdots \times p_{j_k}^{\nu_{j_k}}\rangle \\ = \sqrt{\nu_{j_l}}|n' \equiv n/p_{j_l}\rangle, \end{aligned} \tag{25}$$

proceeding as in Eq. (24) except that we use Eq. (12) instead of Eq. (11).

Using our correspondence rooted trees $\leftrightarrow$ occupation number formalism we get $p_{j_k} \to \pi_{\tau(j_k)} = \pi_{\tau_k}, \times \to *, \nu_{j_l(\tau_l)} \to$ number of times the tree π_{τ_l} appears in $\pi_{\tau_1} * \cdots * \pi_{\tau_d}$. Therefore, we get

$$\hat{\alpha}_{\tau_l}^\dagger(\tau = \pi_{\tau_1} * \cdots * \pi_{\tau_d}) = \sqrt{\nu_{j_l(\tau_l)}+1}\tau * \pi_{\tau_l} \tag{26}$$

and

$$\hat{\alpha}_{\tau_l}(\tau = \tau' * \pi_{\tau_l}) = \sqrt{\nu_{j_l(\tau_l)}}\tau', \tag{27}$$

using the commutativity of $*$. Equations (26) and (27) for rooted trees are the counterparts of Eqs. (24) and (25) in the occupation number formalism, respectively. Just to get used to the meaning of the action of $\hat{\alpha}_{\tau_\ell}^\dagger$ on τ, the term $\tau * \pi_{\tau_l}$ is the tree obtained from τ by attaching the tree π_{τ_l} to the root of τ. The algebra generated by $\{1, \hat{\alpha}_\tau^\dagger, \hat{\alpha}_\tau, \hat{\mathcal{N}}_\tau \equiv \hat{\alpha}_\tau^\dagger\hat{\alpha}_\tau\}$ is the Heisenberg–Weyl algebra built on trees.

5 Concluding remarks

This work unravels the interplay of recurrent constructions on Fock space (Heisenberg–Weyl algebra) and on graded vector spaces with basis elements trees (the grafting operator). We believe this situation is promising, because, on the one hand, it reveals new features of the Hopf algebra itself, e.g., we recognized the important role of the fundamental theorem of arithmetic in the construction of the operators B_+ and B_- once we deal with the occupation number formalism where states are indexed by natural numbers. Recall that Eqs. (18), (19), (20) and (21) are the counterparts of the definitions (5), (6), (7) and (8), respectively. On the other hand, we have shown how to construct the Heisenberg–Weyl algebra, usually described in the Fock space framework, on the space made of rooted trees. We end up by saying that once a dictionary is established between the basic building blocks of Hopf algebras built on graded vector spaces with basis elements rooted trees and the counterparts in the Fock space, one can gain insight on the structure of both pointing towards further explorations and connections. E.g., it would be interesting to unveil other counterparts of rooted trees in the context of Fock space vectors and vice versa. We would like to mention three particular examples. The realization of the operator N (which play an important role in the C–K work of Ref. [13] to establish a connection between the Hopf algebra of renormalization in quantum field theory and the Hopf algebra associated to the computation of the transverse index theory of foliations in the realm of non-commutative geometry) in Fock space should follow from the results developed here. Another topic worth further investigation is the representation of the results in Ref. [30] in the context of Fock space. Also, we believe that it would be of interest to explore, e.g., the construction of generalized Heisenberg–Weyl algebras in the context of graded vector spaces made of rooted trees (for a recent review see Ref. [41]).

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